UDC 539.3:534.1

INFLUENCE OF GEOMETRICAL NONLINEARITY ON THE WAVES PROPAGATING THROUGH A THIN, FREE PLATE

PMM Vol. 43, No. 6, 1979, pp. 1089-1094

L. A. SHENIAVSKII
(Moscow)

(Received November 4, 1978)

Exact wave solutions of the equations of motion of a thin plate are obtained for the one-dimensional case of a first approximation model. The dependence of the velocity of propagation of flexural-longitudinal waves on their frequency is computed for different values of the amplitude of the bending components.

The nonlinear character of the relation connecting the deformations and displacements in the theory of thin shells may give rise to effects which cannot be described in terms of the linear approximation even in those cases when the Hooke's law still holds [1]. The estimation of the magnitude of the displacements at which the effects caused by the geometrical nonlinearity become apparent, is of interest.

1. Let us introduce the Cartesian x, y, z-coordinates in such a manner, that the plane $z = 0$ coincides with the middle surface of the plate. We denote the components of the displacements of the points of the middle surface in the x, y and z directions, by u, v and w, respectively. To simplify the computations, we shall use the first approximation model of the theory of thin shells and consider the case of one-dimensional motion, i.e. we shall regard u, v and w as functions of time t and coordinate x. Assuming that the radii of curvature in the equations of motion of thin hollow shells given in [2] tend to infinity, we obtain

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} - \rho\frac{1-\sigma^2}{E_n}\frac{\partial^2 u}{\partial t^2} = 0 \tag{1.1}$$

$$\frac{1-\sigma}{2}\frac{\partial^2 v}{\partial x^2} - \rho\frac{1-\sigma^2}{E_n}\frac{\partial^2 v}{\partial t^2} = 0$$

$$\frac{h^2}{12}\frac{\partial^4 w}{\partial x^4} - \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial x}\right) + \rho\frac{1-\sigma^2}{E_n}\frac{\partial^2 w}{\partial t^2} = 0$$

where E_n, σ, ρ and h are the Young's modulus, Poisson's ratio, plate material density and its thickness, respectively. Neglecting the nonlinear terms in (1.1), we obtain the equations of the classical theory of plates, the conditions of applicability of which were estimated in [3] by comparing the theoretical dispersion curves with the experimental data. The second equation of the system (1.1) which describes the transverse wave, is independent under the assumptions made. We shall seek the solution for u and w in the form of waves propagating with equal velocities: $c : u = u(\xi)$, $w = w(\xi)$ where $\xi = t - x/c$. Let us put $du/d\xi = X$, $dw/d\xi = Z$ and $c_0^2 = E_n/(\rho(1-\sigma^2))$ and integrate the equations for X and Z. Since the plate sides are physically indistinguishable from each other, therefore the system describing the flexural-longitudinal waves can be written as

$$X = \frac{1}{2ac}(C_1 - Z^2) \tag{1.2}$$

$$\frac{h^2}{12}\frac{d^2Z}{d\xi^2} + cXZ + c^2(a+1)Z = 0 \quad \left(a = \frac{c^2}{c_0^2} - 1\right)$$

Since it can be shown that when $a = 0$ then the bending component $Z = \text{const}$ has only the non-wave solutions, we shall assume that $a \neq 0$.

2. When the physical nonlinearity is taken into account, the equation of longitudinal oscillations of the plate has a solution in the form of propagating isolated impulses (solitons) [4]. In the case under consideration of flexural-longitudinal waves for a similar solution should fulfil the conditions: $X \to 0$, $Z \to 0$ and $dZ/d\xi \to 0$, when $\xi \to \infty$. In this case $C_1 = 0$ in (1.2) and the system has the following solutions for the bending component : $Z = 0$, $Z = Z_0 = 2c\sqrt{a(a+1)}$ and $Z = Z_0 \sec(2\sqrt{3}c^2\xi/(hc_0))$. The conditions at infinity are not satisfied by these solutions. Since the system (1.1) holds for hollow shells it follows that its solutions should have finite derivatives. This implies that the flexural-longitudinal waves have no solutions in the form of the propagating isolated impulses within the framework of the first approximation model.

We shall seek the periodic wave solutions. In this case the first equation of (1.2) implies that $C_1 \geqslant 0$. Eliminating X from the system, we reduce the equation for Z to the form

$$\frac{ah^2}{3}\left(\frac{dZ}{d\xi}\right)^2 = Z^4 - 2AZ^2 + C_2 \quad \left(A = C_1 + 2a\frac{c^4}{c_0^2}\right) \tag{2.1}$$

According to [5], the solution of (2.1) can be expressed in terms of elliptic functions the actual form of which is determined by the number of positive roots of the trinomial of the right hand side of (2.1). Since the functions Z and $dZ/d\xi$ should vanish periodically, if follows from (2.1) that the sign of C_2 is the same as that of a, and the form of the solution of (2.1) depends on the sign of a. When $a > 0$, the expression (2.1) can conveniently be written in the form

$$\frac{ah^2}{3}\left(\frac{dZ}{d\xi}\right)^2 = (\alpha^2 - Z^2)(\beta^2 - Z^2) \quad (\alpha > \beta)$$

According to [5, 6] we have

$$Z = \beta\,\mathrm{sn}\,(\Omega\xi), \quad \Omega = (\alpha/h)/\sqrt{a/3} \tag{2.2}$$

Here sn denotes the elliptic sine and $k = \alpha/\beta$ is its modulus. Substituting the value of Z obtained into (1.2), we obtain X. Integrating the expression for X, we separate the oscillatory part expressing it by the Jacobi zeta function zn [6, 7]. At the same time we must assume, in order that the function u be periodic, that

$$C_1 = \alpha^2(1 - \mathbf{E}/\mathbf{K}) \tag{2.3}$$

where $\mathbf{K}$ and $\mathbf{E}$ are complete elliptic integrals of first and second kind, respectively. When the relation (2.3) holds, we have

$$u = \frac{\alpha^2}{2ac\Omega} \operatorname{zn}(\Omega\xi) \tag{2.4}$$

Using (2.2) and choosing the integration constant appropriately, we can write the expression for w in the following form symmetrical with respect to $w = 0$:

$$w = h\sqrt{\frac{a}{3}} \ln \frac{\operatorname{dn}\eta - k \operatorname{cn}\eta}{\sqrt{1-k^2}}, \quad \eta = \Omega\xi \tag{2.5}$$

$$\left(w_0 = \max w = \frac{1}{2} h \sqrt{\frac{a}{3}} \ln \frac{1+k}{1-k}\right)$$

Here cn and dn denote the Jacobi functions, namely the elliptic cosine and the delta of the amplitude. The quantity w_0 represents the amplitude of the bending component of w.

Since the periods of the elliptic functions are expressed in terms of the complete elliptic integral of the first kind $\mathbf{K}(k)$, it is expedient to replace Ω by $\nu = \Omega/(4\mathbf{K})$. Taking (2.1) and (2.3) into account and using the relations connecting the quantities α and β, we can express them in terms of the modulus k. Thus, taking into account (2.2), (2.4) and (2.5), we can write the wave solutions of the system (1.1) for the case $c > c_0$, describing the flexural-longitudinal oscillations, in the form

$$w(\xi) = h\sqrt{\frac{a}{3}} \ln \frac{\operatorname{dn}(4\mathbf{K}\nu\xi) - k \operatorname{cn}(4\mathbf{K}\nu\xi)}{\sqrt{1-k^2}}, \quad u(\xi) = \frac{2h^2\mathbf{K}\nu}{3c} \operatorname{zn}(4\mathbf{K}\nu\xi) \tag{2.6}$$

$$\nu = \frac{\sqrt{3}c^2}{2hc_0} \frac{1}{\sqrt{\mathbf{K}(\mu)\mathbf{E}(\mu)}}, \quad \mu = \frac{2\sqrt{k}}{1+k}$$

In order to estimate the degree of nonlinearity of the wave, we expand w into a Fourier series. Using the trigonometric series given in [6] for the function sn and integrating this series term by term, we obtain

$$w_{0n} = 4h\sqrt{\frac{a}{3}} \frac{q^{n-1/2}}{(2n-1)(1-q^{2n-1})}, \quad q = \exp\left(-\pi \frac{\mathbf{K}'}{\mathbf{K}}\right)$$

where w_{0n} is the n-th harmonic of $w(\xi)$, $\mathbf{K}' = \mathbf{K}(k')$, $k' = \sqrt{1-k^2}$ is the supplementary modulus. This gives, in particular, the following expression for the ratio of the first and second harmonics:

$$\gamma_{12} = w_{01}/w_{02} = 3\,[1 + 2 \operatorname{ch}(\pi\mathbf{K}'/\mathbf{K})]$$

The following relations also hold:

$$\lim_{k\to 0} \gamma_{12} = \infty, \quad \lim_{k\to 1} \gamma_{1n} = (2n-1)^2$$

where γ_{1n} is the ratio of the first and n-th harmonic. Thus, when k varies from zero to one, the form of the wave,component of w changes from the harmonic to the triangular.

The following dimensionless quantities are useful in the numerical analysis of the relations (2.6):

$$V = c/c_0, \quad \zeta_0 = \sqrt{3}\, w_0/h, \quad \nu_0 = 2h\nu/(\sqrt{3}\, c_0) \tag{2.7}$$

Fig. 1 depicts the characteristic profiles of the components u and w, and the displacement profile (dashed line) obtained on a digital computer for $\zeta_0 = 6.5$ and $\nu_0 = 3.7$ $(k = 0.98)$. The values of ξ expressed as wavelengths are plotted on the abscissa. The values of w are reduced by the factor of about 0.76, and the ordinates of the curve $u(\xi)$ are increased by about 1.5 times in order to improve the clarity of the representation. The deformation vectors $\{u_i, w_i\}$ illustrating the displacement of the points $\xi = \xi_i$ of the middle surface are constructed for these points.

If we fix ζ_0 and reduce ν_0 (or increase ζ_0 for a fixed ν_0), then the profile of the component w will become more triangular, the component u will decrease and the displacement profile will approach the profile of the component w. When ν_0 is increased, the profile of w tends to the harmonic shape, the component u increases and the displacement profile changes, first tending to the Π-shape, and then to the Ω-shape. When the component u is increased still further, the equations begin to describe self-overlapping displacement profiles which are not physically realizable. When ν_0 is fixed and ζ_0 decreases, the component u also decreases and the profiles of the bending component and of the displacement also approach the harmonic profiles.

In the case $a < 0$ we see from (2.1) that $C_2 < 0$. Let us write the equation for Z in the form

$$\frac{-ah^2}{3}\left(\frac{dZ}{d\xi}\right)^2 = (\alpha^2 + Z^2)(\beta^2 - Z^2)$$

Arguments and computations analogous to those given above, show that the solutions of (1.1) can be written for the flexural-longitudinal waves with $c < c_0$, in the form

$$w(\xi) = h\sqrt{\frac{-a}{3}} \arcsin[k \operatorname{sn}(4\mathrm{K}\nu\xi)], \quad u(\xi) = \frac{2h^2\mathrm{K}\nu}{3c} \operatorname{zn}(4\mathrm{K}\nu\xi) \qquad (2.8)$$

$$\nu = \frac{\sqrt{3}c^2}{2hc_0}\{\mathrm{K}(k)[2\mathrm{E}(k) - \mathrm{K}(k)]\}^{-1/2}$$

The expression for the amplitude of the n-th harmonic of the component w has the following form for $a < 0$:

$$w_{0n} = 4h\sqrt{\frac{-a}{3}} \frac{q^{n-1/2}}{(2n-1)(1+q^{2n-1})}$$

and from this we have

$$\gamma_{12} = 3[2 \operatorname{ch}(\pi\mathrm{K}'/\mathrm{K}) - 1] \quad (\lim_{k\to 0}\gamma_{12} = \infty, \ \lim_{k\to 1}\gamma_{1n} = 2n - 1)$$

consequently, when k varies from zero to one, the waveform of the component w is transformed from the harmonic to the square shape. The modulus k is bounded in all expressions corresponding to the case $a < 0$ by the quantity k_* where $k_* \approx 0.91$ is a root of the equation $2\mathrm{E}(k) = \mathrm{K}(k)$.

The characteristic profiles of the components w and u and the displacement profile are shown for the case $c < c_0$ in Fig. 2, which is analogous to Fig. 1, for $\zeta_0 \approx 0.97$ and $\nu_0 \approx 0.66$ $(k = 0.9)$. The values of w are increased by the factor of 3.56 and the values of the ordinates $u(\xi)$ by about two times.

If we keep ν_0 fixed and increase ζ_0 (or keep ζ_0 fixed and increase ν_0) then the profiles of w and u will not change appreciably, the values of u will increase and the displacement profile will tend to the triangular one. When ζ_0 (or ν_0) are further increased, the displacement profile becomes self-intersecting. When ζ_0 (or ν_0) decreases, the profile of w tends to the harmonic, the component u diminishes and the displacement profile tends to the profile of w.

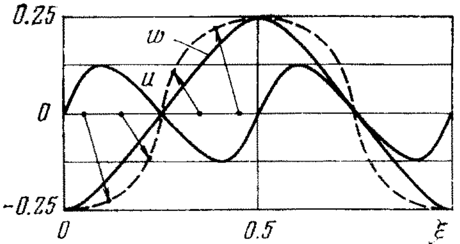

Fig. 1

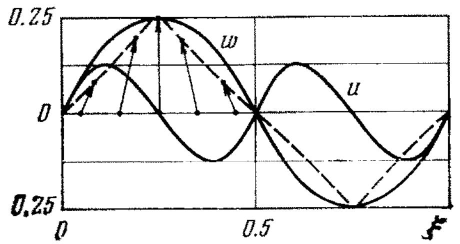

Fig. 2

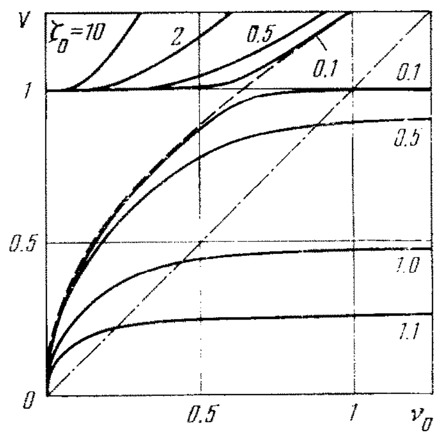

Fig. 3

Thus, depending on the velocity of propagation, the system (1.1) admits solutions describing two types of coupled flexural-longitudinal waves. The period of the zeta function is twice as small as the period of the functions sn and cn, therefore from (2.6) and (2.8) it follows that the oscillation frequency of the component u is twice that of the bending components. These coupled waves are fully defined by two parameters such as e. g. the velocity of propagation and the modulus k.

3. Since w is the most easily measured quantity, it follows that in studying the dependence of the velocity of propagation on the frequency it is expedient to use the amplitude of the bending component w_0 as the parameter. Using the dimensionless quantities (2.7) we obtain from (2.5), (2.6) and (2.8)

$$\zeta_0 = \begin{cases} (V^2 - 1)^{1/2} \operatorname{Ar\,th}(k), & V > 1 \\ (1 - V^2)^{1/2} \arcsin(k), & V < 1 \end{cases} \tag{3.1}$$

$$\nu_0 = \begin{cases} V^2 \{K(\mu) E(\mu)\}^{-1/2}, & V > 1 \\ V^2 \{K(k)[2E(k) - K(k)]\}^{-1/2}, & V < 1 \end{cases}$$

Fig. 3 depicts the dependence of V on ν_0 for various values of ζ_0 computed on a digital computer. The curves shown in the graph are analogs of the dispersion curves for a nonlinear plate. The dashed line depicts the dispersion curve for a linear plate the equation of which, written in terms of the variables V and ν_0, has the form $V^2 = \pi\nu_0/2$. The equations of the theory of shells hold when the length of the expanding wave λ is greater than h. Introducing the dimensionless wavelength $\Lambda = \sqrt{3}\,\lambda/(2\,h)$, we can write this condition as $\Lambda < 1$. The straight line for which $\Lambda = 1$ is given in dash-dot.

The curves situated below the straight line $V = 1$ correspond to a type of waves with a Π-shaped profile of the bending component, and for these waves we have $\zeta_0 < \zeta_*$ where $\zeta_* = \arcsin k_* = 1.1433$. The curves have horizontal asymptotes defined by the formula $V = \sqrt{1 - (\zeta_0 / \zeta_*)^2}$. The quantity k increases smoothly along the curve with increasing V, and tends asymptotically to k_*. The component u increases together with it in accordance with (2.8), and becomes predominant when k approaches k_*, for any values of ζ_0.

The curves situated above the straight line $V = 1$ describe the waves with a triangular component w. To find their asymptotic behavior when k approaches unity, we use the expansion of $\mathbf{K}$ in powers of the auxilliary modulus. In particular, taking into account only the first term of the expansion we obtain $\mathbf{K}(\mu) \approx 2\,(\ln 2 + \zeta_0 / \sqrt{a})$, and using the same approximation we find that $\mathbf{E}(\mu) \approx 1$. Substituting these values into (3.1), we obtain the following asymptotic formula for $\zeta_0 / \sqrt{a} \gg \ln 2$:

$$\nu_0 \approx V^2 (V^2 - 1)^{1/4} (2\zeta_0)^{-1/2} \tag{3.2}$$

Computations show that when $\zeta_0 < 2$, then formula (3.2) holds only when $V \approx 1$ ($V \leqslant 1.01$ when $\zeta_0 = 0.5$). For $\zeta_0 \geqslant 10$ when $V = 1.5$, the error of estimating ν_0 by means of (3.2) does not exceed 4% and becomes smaller with diminishing V.

Let us now find the limiting value of the distribution coefficient $\varkappa = w_0/u_0$ where u_0 is the amplitude of the component u. Using (2.6) and (3.1),we write $\varkappa$ in the form

$$\varkappa = \frac{(1+k)\,\zeta_0}{V\,\mathrm{zn}_0}\sqrt{\frac{\mathbf{E}(\mu)}{\mathbf{K}(\mu)}} \tag{3.3}$$

where zn_0 is the amplitude of the zeta function. Using the relations given in [5, 7], we can show that $\mathrm{zn}_0 \to 1$ as $k \to 1$. Substituting $\mathrm{zn}_0 = 1$ into (3.3) and passing to the limit as $k \to 1$, we find that $\varkappa \to \sqrt{2\,\xi_0\sqrt{a}/V}$. This in particular implies that when $V \to 1$, then $\varkappa \to 0$, , i.e. the component u becomes predominant. It should be noted that when $k \to 1$, the minimum value of the radius of curvature of the profile of w tends to zero. If λ does not tend to infinity at the same time sufficiently rapidly, then the theory of hollow shells ceases to be applicable.

From the graphs in Fig. 3 we see that when $\zeta_0 \leqslant 0.1$, the curves almost coincide with the dispersion curve of the linear system everywhere except in the region in which the velocity c is almost equal to c_0. Within this region the waves of both types differ appreciably from the linear wave. However, when $c \to c_0$, the nonlinear waves degenerate into a purely longitudinal wave satisfying the system (1.1) at $w \equiv 0$.

Comparison of the dispersion curves of the classical theory of plates with the experimental data [3] shows that in the case of frequencies for which $c > c_t = c_0 \sqrt{(1-\sigma)/2}$, the inertia of rotation of the plate element plays a significant part. This inertia is not, however, taken into account in deriving the initial equations (1.1). The influence of the rotational inertia can alter appreciably the form of the curves corresponding to the case $c > c_0$.

REFERENCES

1. Novozhilov, V. V. Foundations of the Non-linear Theory of Elasticity. Rochester, New York, Graylock Press, 1953.

2. Vol'mir, A. S., Nonlinear Dynamics of Plates and Shells. Moscow "Nauka", 1972.

3. Skuchik, E. Simple and Complex Oscillatory Systems. Moscow, "Mir", 1971.

4. Ostrovskii, L. A. and Sutin, A.M., Nonlinear elastic waves in rods. PMM Vol. 41, No. 3, 1977.

5. Akhiezer, N. I. Elements of the Theory of Elliptic Functions. Moscow, "Nauka", 1970.

6. Gradshtein, I. S. and Ryzhik, I. M. Tables of Integrals, Sums, Series and Products. Moscow, Fizmatgiz, 1963.

7. Jahnke, E., Emde, F. and Lösch, F. Tables of Higher Functions. 6th rev. ed. N.Y. McGraw — Hill, 1960.

Translated by L. K.